COSC-6590/GSCS-6390

Games: Theory and Applications

Lecture 06 - Computation of Mixed Saddle-Point Equilibrium Policies

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Graphical Method

Graphical Method

To find **mixed saddle-point equilibria**
- compute **mixed security policies** for both players

For 2×2 games we can use the **graphical method**

$$A = \underbrace{\begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices}$$

Compute the mixed security policy for P_1

$$\min_{y=[y_1 \quad y_2]} \max_{z=[z_1 \quad z_2]} y'Az = \min_y \max_z [y_1 \quad y_2] \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
$$= \min_y \max_z z_1(3y_1 - y_2) + z_2(y_2)$$
$$= \min_y \max\{3y_1 - y_2, y_2\}$$

Graphical Method

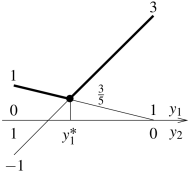
Since $y_1 + y_2 = 1$, we must have $y_2 = 1 - y_1$ and therefore

$$\min_y \max_z y' A z = \min_{y_1 \in [0,1]} \max\{4y_1 - 1, 1 - y_1\}.$$

To find the optimal value for y_1

- draw the two lines $4y_1 - 1$ and $1 - y_1$ in the same axis
- pick the maximum point-wise
- select the point y_1^* for which the maximum is smallest.

Point is the security policy. Maximum is the value of the game.



$$\min_y \max_z y' A z = \frac{3}{5},$$

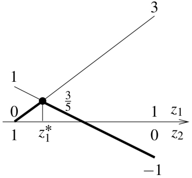
$$y_1^* = \frac{2}{5} \Rightarrow y^* = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \end{bmatrix}'$$

Graphical Method

Compute the mixed security policy for P_2

$$\max_{z=[z_1 \quad z_2]} \min_{y=[y_1 \quad y_2]} y' A z = \max_z \min_y [y_1 \quad y_2] \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
$$= \max_z \min_y y_1(3z_1) + y_2(-z_1 + z_2)$$
$$= \max_z \min\{3z_1, -z_1 + z_2\}$$

This results in



$$\max_z \min_y y' A z = \frac{3}{5},$$

$$z_1^* = \frac{1}{5} \Rightarrow z^* = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix}'$$

Linear Program Solution

Linear Program Solution

Systematic numerical procedure to find mixed saddle-point equilibria.

Goal is to compute

$$V_m(A) = \min_{y \in \mathcal{Y}} \max_{z \in \mathcal{Z}} y' A z = \max_{z \in \mathcal{Z}} \min_{y \in \mathcal{Y}} y' A z$$

where

$$\mathcal{Y} := \left\{ y \in \mathbb{R}^m : \sum_i y_i = 1, \quad y_i \geq 0, \quad \forall i \right\}$$

$$\mathcal{Z} := \left\{ z \in \mathbb{R}^n : \sum_j z_j = 1, \quad z_j \geq 0, \quad \forall j \right\}$$

Linear Program Solution

Compute the **inner max** in the $\min_y \max_z$ optimization.

For a fixed $y \in \mathcal{Y}$ we have

$$\max_{z \in \mathcal{Z}} y'Az = \max_{z \in \mathcal{Z}} \sum_{ij} y_i a_{ij} z_j = \max_{z \in \mathcal{Z}} \sum_j \left(z_j \sum_i y_i a_{ij} \right) = \max_j \sum_i y_i a_{ij}$$

Use an equality to convert a maximization into a constrained minimization: given a set of numbers $x_1, x_2, \ldots, x_m$,

$$\max_j x_j = \min \{ v \in \mathbb{R} : v \geq x_j, \quad \forall j \}$$

Using in the previous equation we conclude that

$$\max_{z \in \mathcal{Z}} y'Az = \min \left\{ v : v \geq \underbrace{\sum_i y_i a_{ij}}_{\substack{j\text{th entry of row vector } y'A, \text{ or} \\ j\text{th entry of column vector } A'y}} , \forall j \right\}$$

Linear Program Solution

Denoting by **1** a column vector consisting of ones, we re-write the condition in the set above as

$$v\mathbf{1} \dot{\geq} \begin{bmatrix} \sum_i y_i a_{i1} \\ \sum_i y_i a_{i2} \\ \vdots \\ \sum_i y_i a_{in} \end{bmatrix} = A'y$$

This allows us to re-write $V_m(A)$ as a **linear program**

$$V_m(A) = \min_{y \in \mathcal{Y}} \min\{v : v\mathbf{1} \dot{\geq} A'y\}$$

$$= \underbrace{\begin{array}{ll} \text{minimum} & v \\ \text{subject to} & \left.\begin{array}{l} y \dot{\geq} 0 \\ \mathbf{1}y = 1 \end{array}\right\} \quad y \in \mathcal{Y} \\ & A'y \dot{\leq} v\mathbf{1} \end{array}}_{\text{optimization over } m+1 \text{ parameters } (v, y_1, y_2, \ldots, y_m)}$$

Linear Program Solution

MATLAB® Hint 2.

Linear programs can be solved numerically with matlab using `linprog` from the Optimization toolbox or the freeware Disciplined Convex Programming toolbox, also known as CVX.

Solving this optimization, we obtain the value of the game v^* and a mixed security policy y^* for P_1.

Since the security policies are those that achieve the minimum in $V_m(A)$, once we have the value of the game we can obtain the set of all mixed security policies using

$$\{y \in \mathbb{R}^m : y \dot{\geq} 0, \quad \mathbf{1}'y = 1, \quad v^*\mathbf{1} \dot{\geq} A'y\}$$

Linear Program Solution

Focusing on the $\max_z \min_y$ optimization, we conclude that

$$\min_{y \in \mathcal{Y}} y' A z = \cdots = \max\{v : v\mathbf{1} \dot{\leq} Az\}$$

Therefore

$$V_m(A) = \underbrace{\begin{array}{ll} \text{maximum} & v \\ \text{subject to} & \left.\begin{array}{l} z \dot{\geq} 0 \\ \mathbf{1}z = 1 \end{array}\right\} \; z \in \mathcal{Z} \\ & Az \dot{\geq} v\mathbf{1} \end{array}}_{\text{optimization over } n+1 \text{ parameters } (v, z_1, z_2, \ldots, z_n)}$$

Solving this optimization, we obtain the value of the game v^* and a mixed security policy z^* for P_2.

And we can obtain the set of all mixed security policies using

$$\left\{z \in \mathbb{R}^n : z \dot{\geq} 0, \;\; \mathbf{1}' z = 1, \;\; v^* \mathbf{1} \dot{\leq} Az\right\}$$

Linear Programs with MATLAB

Linear Programs with MATLAB

MATLAB® Hint 2. (Linear programs).

`[x,val ] = linprog (c,Ain,bin,Aeq,beq,low,high)`

from MATLAB®'s Optimization Toolbox numerically solves linear programs of the form

$$\begin{array}{ll} \text{minimum} & \texttt{c'x} \\ \text{subject to} & \texttt{Ain x} \dot{\leq} \texttt{ bin} \\ & \texttt{Aeq x} = \texttt{beq} \\ & \texttt{low} \dot{\leq} x \dot{\leq} \texttt{high} \end{array}$$

`val`: value of the minimum.

`x`: vector that achieves the minimum

To avoid the corresponding inequality constraints
- the vector `low` can have some or all entries equal to `-Inf`
- the vector `high` can have some or all entries equal to `Inf`

Linear Programs with MATLAB

Same optimization performed with the Disciplined Convex Programming (CVX) Toolbox

```
cvx_begin
  variables x(size(Ain,2))
  minimize c*x
  subject to
    Ain*x <= bin;
    Aeq*x = beq;
    x >= low;
    x <= high;
cvx_end
```

CVX syntax is especially intuitive.

Linear Programs with MATLAB

CVX code to find the value of a game defined by a matrix `A`
- and the mixed value and security policy for P_1 (left)
- and the mixed value and security policy for P_2 (right)

```
cvx_begin
  variables v y(size(A,1))
  minimize v
  subject to
    y >= 0;
    sum(y) == 1;
    A'*y <= v;
cvx_end
```

```
cvx_begin
  variables v z(size(A,2))
  maximize v
  subject to
    z >= 0;
    sum(z) == 1;
    A*z >= v;
cvx_end
```

Strictly Dominating Policies

Strictly Dominating Policies

Consider a game specified by an $m \times n$ matrix A.

- m actions for P_1, and n actions for P_2.

$$A = \underbrace{\begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{bmatrix}}_{P_2 \text{ choices (maximizer)}} \Bigg\} P_1 \text{ choices (minimizer)}$$

Strictly Dominating Policies

We say that row i strictly dominates row k if

$$a_{ij} < a_{kj} \quad \forall j$$

which means that no matter what P_2 does, the minimizer P_1 is always better off by selecting row i instead of row k.

In practice, this means that

- Pure policies: P_1 will never select row k
- Mixed policies: P_1 will always select row k with probability zero, i.e., $y_k^* = 0$ for any security/saddle-point policy.

Strictly Dominating Policies

Conversely, we say that column j strictly dominates column l if

$$a_{ij} > a_{il} \qquad \forall i$$

which means that no matter what P_1 does, the maximizer P_2 is always better off by selecting column j instead of column l.

In practice, this means that

- Pure policies: P_2 will never select column l
- Mixed policies: P_1 will always select col l with probability zero, i.e., $z_l^* = 0$ for any security/saddle-point policy.

Strictly Dominating Policies

Finding dominating rows/columns in A allows one to reduce the size of the problem that needs to be solved, as we can:

1. remove any rows/columns that are strictly dominated
2. compute (pure or mixed) saddle-point equilibria for the smaller game
3. recover the saddle-point equilibria for the original:
   - pure policies: saddle-point equilibria are the same, modulo some re-indexing to account for the fact that indexes of the rows/columns may have changed
   - mixed policies: may need to insert zero entries corresponding to the columns/rows that were removed.

By removing strictly dominated rows/columns one cannot lose security policies so all security policies for the original (larger) game correspond to security policies for the reduced game.

Strictly Dominating Policies

Example 6.1 (Strictly dominating policies).

$$A = \underbrace{\left.\begin{bmatrix} 3 & -1 & 0 & -1 \\ 4 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}\right\} P_1 \text{ choices (minimizer)}}_{P_2 \text{ choices (maximizer)}}$$

Since the 2nd row is strictly dominated by the 1st row

$$A^{\dagger} = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

We now observe that both the 2nd and 4th column are (strictly) dominated by the 3rd column

$$A^{\ddagger} = \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}$$

Strictly Dominating Policies

We found the following value and mixed security/saddle-point equilibrium policies

$$V(A^{\ddagger}) = \frac{3}{5}, \qquad y^* = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \end{bmatrix}, \qquad z^* = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix},$$

We thus conclude that the original game has the following value and mixed security/saddle-point equilibrium policies

$$V(A) = \frac{3}{5}, \qquad y^* = \begin{bmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \qquad z^* = \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{4}{5} \\ 0 \end{bmatrix},$$

“Weakly” Dominating Policies

"Weakly" Dominating Policies

We say that row i (weakly) dominates row k if

$$a_{ij} \leq a_{ik} \qquad \forall j$$

which means that no matter what P_2 does, the minimizer P_1 loses nothing by selecting row i instead of row k.

We say that column j (weakly) dominates column l if

$$a_{ij} \geq a_{il} \qquad \forall l$$

which means that no matter what P_1 does, the maximizer P_2 loses nothing by selecting column j instead of column l.

"Weakly" Dominating Policies

Remove weakly dominated rows/columns and be sure that

- the value of the reduced game is the same as the value of the original game, and
- one can reconstruct security policies/saddle-point equilibria for the original game from those for the reduced game.

One may lose some security policies/saddle-point equilibria that were available for the original game but that have no direct correspondence in the reduced game.

A game is said to be **maximally reduced** when no row or column dominates another one.

Saddle-point equilibria of maximally reduced games are called **dominant**.

"Weakly" Dominating Policies

Example 6.2 (Weakly dominating policies).

$$A = \underbrace{\left.\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\right\} P_1 \text{ choices}}_{P_2 \text{ choices}}$$

Game has value $V(A) = 1$ and two pure saddle-point equilibria $(1, 2)$ and $(2, 2)$. However, this game can be reduced as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\text{(remove latter)}]{\text{2nd row dominates 1st}} A^{\dagger} := \begin{bmatrix} -1 & 1 \end{bmatrix} \xrightarrow[\text{(remove latter)}]{\text{2nd col dominates 1st}} A^{\ddagger} := \begin{bmatrix} 1 \end{bmatrix}$$

from which one obtains the pure saddle-point equilibrium $(2, 2)$.

- i.e., the $\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ pair of mixed policies.

"Weakly" Dominating Policies

Alternatively, this game can also be reduced as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow[\text{(remove latter)}]{\text{2nd col dominates 1st}} A^{\dagger} := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow[\text{(remove latter)}]{\text{1st row dominates 2nd}} A^{\ddagger} := [1]$$

from which one obtains the pure saddle-point equilibrium $(1, 2)$.

- i.e., the $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ pair of mixed policies.

Practice Exercises

Practice Exercises

6.1 (Mixed security levels/policies - graphical method).
For the two zero-sum matrix games compute the average security levels and all mixed security policies for both players.

$$A = \underbrace{\begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices} \qquad B = \underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}}_{P_2 \text{ choices}} \Bigg\} P_1 \text{ choices}$$

$$C = \underbrace{\begin{bmatrix} 1 & 3 & -1 & 2 \\ -3 & -2 & 2 & 1 \\ 0 & 2 & -2 & 1 \end{bmatrix}}_{P_2 \text{ choices}} \Bigg\} P_1 \text{ choices} \qquad D = \underbrace{\begin{bmatrix} 2 & 1 & 0 & -1 \\ -1 & 3 & 1 & 4 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices}$$

Use the graphical method.

Hint: for 3×2 and 2×3 games start by computing the average security policy for the player with only two actions.

Practice Exercises

Solution for the Matrix A

$$V_m(A) = \min_y \max_z \quad y_1 z_1 + 4y_1 z_2 + 3y_2 z_1 - y_2 z_2$$

$$= \min_y \max\{y_1 + 3y_2, 4y_1 - y_2\} = \min_y \max\{-2y_1 + 3, 5y_1 - 1\} = \frac{13}{7}$$

with the (unique) mixed security policy for $P_1 : y^* := \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \end{bmatrix}'$, and

$$V_m(A) = \max_z \min_y \quad y_1 z_1 + 4y_1 z_2 + 3y_2 z_1 - y_2 z_2$$

$$= \max_z \min\{z_1 + 4z_2, 3z_1 - z_2\} = \max_z \min\{-3z_1 + 4, 4z_1 - 1\} = \frac{13}{7}$$

with the (unique) mixed security policy for $P_2 : z^* := \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \end{bmatrix}'$.

Consequently, this game has a single mixed saddle-point equilibrium $(y^*, z^*) = \left(\begin{bmatrix} \frac{4}{7} & \frac{3}{7} \end{bmatrix}', \begin{bmatrix} \frac{5}{7} & \frac{2}{7} \end{bmatrix}' \right)$.

Practice Exercise

Solution for the Matrix B

$$V_m(B) = \max_z \min_y \quad 4y_1 z_1 + 2y_2 z_2 + 3y_3 z_1 + y_3 z_2$$

$$= \max_z \min\{4z_1, 2z_2, 3z_1 + z_2\} = \max_z \min\{4z_1, 2 - 2z_1, 2z_1 + 1\} = \frac{4}{3}$$

with the sole mixed security policy for $P_2 : z^* := \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}'$.

P_1 has more than 2 actions:

- cannot use the graphical method to find the mixed security policies for this player.

Practice Exercise

However, from

$$\left\{ y \in \mathbb{R}^m : y \dot{\geq} 0, \quad \mathbf{1}' y = 1, \quad v^* \mathbf{1} \dot{\geq} A' y \right\}$$

we know that the mixed security policies for P_1 must satisfy

$$\frac{4}{3}\mathbf{1} \dot{\geq} A'y = \begin{bmatrix} 4y_1 + 3y_3 \\ 2y_2 + y_3 \end{bmatrix} = \begin{bmatrix} y_1 - 3y_2 + 3 \\ -y_1 + y_2 + 1 \end{bmatrix}$$

$$\Leftrightarrow y_1 \leq -\frac{5}{3} + 3y_2, y_1 \geq y_2 - \frac{1}{3}, (y_1 \leq 1 - y_2)$$

which has a single solution $y^* := [\frac{1}{3} \quad \frac{2}{3} \quad 0]'$.

Consequently, game has a single mixed saddle-point equilibrium

$$(y^*, z^*) = \left(\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}', \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}' \right)$$

Practice Exercise

Solution for the Matrix C

- Row 3 strictly dominates over row 1.
- Column 2 strictly dominates over column 1.

We can therefore reduce the game to

$$C^{\dagger} := \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

For this matrix

$$\begin{aligned} V_m(C^{\dagger}) &= \min_y \max\{-2y_1 + 2y_2, 2y_1 - 2y_2, y_1 + y_2\} \\ &= \min_y \max\{-4y_1 + 2, 4y_1 - 2, 1\} = 1 \end{aligned}$$

which has multiple minima for $y_1 \in \left[\frac{1}{4} \quad \frac{3}{4}\right]$. These correspond to the following security policies for P_1 in the original game:

$$y^* := [0 \quad y_1 \quad 1 - y_1]' \qquad \text{for any} \qquad y_1 \in \left[\frac{1}{4} \quad \frac{3}{4}\right]$$

Practice Exercise

P_2 has more than 2 actions even for the reduced game $C^\dagger$

- cannot use the graphical method to find the mixed security policies for this player.

However, from $\left\{z \in \mathbb{R}^n : z \dot{\geq} 0, \ \ \mathbf{1}'z = 1, \ \ v^*\mathbf{1} \dot{\leq} Az\right\}$
we know that these policies must satisfy

$$\mathbf{1}\dot{\leq}Az = \begin{bmatrix} -2z_1 + 2z_2 + z_3 \\ 2z_1 - 2z_2 + z_3 \end{bmatrix} = \begin{bmatrix} -3z_1 + z_2 + 1 \\ z_1 - 3z_2 + 1 \end{bmatrix}$$

$$\Leftrightarrow z_2 \geq -3z_1, z_2 \leq \frac{1}{3}z_1, (z_2 \leq 1 - z_1)$$

which has a single solution $z_1 = z_2 = 0$, and corresponds to the security policy for P_2 in the original game: $z^* := [0 \ \ 0 \ \ 0 \ \ 1]'$.

This game has the family of mixed saddle-point equilibria

$$\left(y^*, z^* = \left([0 \ \ y_1 \ \ 1 - y_1]', [0 \ \ 0 \ \ 0 \ \ 1]'\right), \qquad y_1 \in \left[\frac{1}{4} \ \ \frac{3}{4}\right]\right.$$

Practice Exercise

Solution for the Matrix D

- Column 2 strictly dominates over column 3.

We can therefore reduce the game to

$$D^\dagger := \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & 4 \end{bmatrix}$$

For this matrix

$$V_m(D^\dagger) = \min_y \max\{2y_1 - y_2, y_1 + 3y_2, 4y_2 - y_1\}$$
$$= \min_y \max\{3y_1 - 1, 3 - 2y_1, 4 - 5y_1\} = \frac{7}{5}$$

which has a single minimum for $y_1 = \frac{4}{5}$. This corresponds to the security policy for P_1 in the original game: $y^* := [\frac{4}{5} \quad \frac{1}{5}]'$.

Since P_2 has more than 2 actions even for $D^\dagger$, we cannot use the graphical method to find the mixed security policies.

Practice Exercise

However, from $\left\{z \in \mathbb{R}^n : z \dot{\geq} 0, \quad \mathbf{1}'z = 1, \quad v^* \mathbf{1} \dot{\leq} Az\right\}$
we know that these policies must satisfy

$$\frac{7}{5}\mathbf{1} \dot{\leq} Az = \begin{bmatrix} 2z_1 + z_2 - z_3 \\ -z_1 + 3z_2 + 4z_3 \end{bmatrix} = \begin{bmatrix} 3z_1 + 2z_2 - 1 \\ -5z_1 + 3z_2 + 4 \end{bmatrix}$$

$$\Leftrightarrow z_2 \geq -\frac{3}{2}z_1 + \frac{6}{5}, z_2 \leq -5z_1 + \frac{13}{5}, (z_2 \leq 1 - z_1)$$

which has a single solution $z_1 = \frac{2}{5}$, $z_2 = \frac{3}{5}$. This corresponds to the security policy for P_2 in the original game:
$z^* := [\frac{2}{5} \;\; \frac{3}{5} \;\; 0 \;\; 0]'$.

This game has the unique mixed saddle-point equilibrium

$$(y^*, z^*) = \left(\begin{bmatrix} \frac{4}{5} & \frac{1}{5} \end{bmatrix}', \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 \end{bmatrix}' \right)$$

Practice Exercise

6.2 (Mixed security levels/policies - LP method).

For each of the following two zero-sum matrix games compute the average security levels and a mixed security policy

$$A = \underbrace{\left.\begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}\right\} P_1 \text{ choices}}_{P_2 \text{ choices}} \qquad B = \underbrace{\left.\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}\right\} P_1 \text{ choices}}_{P_2 \text{ choices}}$$

Solve this problem numerically using MATLAB®.

Practice Exercise

Solution for the matrix A

Use the CVX code to compute the mixed value and the security policies for P_2 and P_1:

```
A =[3,1;2,2;1,3];
```

```
cvx_begin
  variables v z(size(A,2));
  maximize v;
  subject to
    z>=0;
    sum(z)==1;
    A*z>= v;
cvx_end
```

```
cvx_begin
  variables v y(size(A,1));
  miniimize v;
  subject to
    y>=0;
    sum(y)==1;
    A'*y <= v;
cvx_end
```

Practice Exercise

Code resulted in the mixed value $V_m(A) = 2$ and a saddle-point

$$(y^*, z^*) = \left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}', \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}' \right)$$

However, this matrix has multiple saddle-point equilibria and for y^* so you may get any distribution of the form

$$[\lambda \;\; 1 - 2\lambda \;\; \lambda]', \;\; \forall \lambda \in \left[0, \frac{1}{2}\right]$$

Practice Exercise

Solution for the matrix B

Use the CVX code to compute the mixed value and the security policies for P_2 and P_1:

```
A =[0,1,2,3;1,0,1,2;0,1,0,1;-1,0,1,0];
```

```
cvx_begin
  variables v z(size(A,2));
  maximize v;
  subject to
    z>=0;
    sum(z)==1;
    A*z>= v;
cvx_end
```

```
cvx_begin
  variables v y(size(A,1));
  miniimize v;
  subject to
    y>=0;
    sum(y)==1;
    A'*y <= v;
cvx_end
```

Practice Exercise

Code resulted in the mixed value $V_m(B) = \frac{1}{2}$ and a saddle-point

$$(y^*, z^*) = \left(\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}', \begin{bmatrix} 0 & 0.1858 & \frac{1}{2} & 0.3142 \end{bmatrix}' \right)$$

However, this matrix has multiple saddle-point equilibria and for z^* so you may get any distribution of the form

$$\begin{bmatrix} 0 & \frac{\lambda}{2} & \frac{1}{\lambda} & \frac{1-\lambda}{2} \end{bmatrix}', \quad \forall \lambda \in [0,1]$$

End of Lecture

06 - Computation of Mixed Saddle-Point Equilibrium Policies

Questions?