COSC-6590/GSCS-6390

# Games: Theory and Applications

## Lecture 10 - Computation of Nash Equilibria for Bimatrix Games

Luis Rodolfo Garcia Carrillo

School of Engineering and Computing Sciences
Texas A&M University - Corpus Christi, USA

# Table of contents

# Completely Mixed Nash Equilibria

# Completely Mixed Nash Equilibria

Alternative version of the battle of the sexes game

$$A = \underbrace{\left.\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}\right\}}_{P_2 \text{ choices}} P_1 \text{ choices} \qquad B = \underbrace{\left.\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\right\}}_{P_2 \text{ choices}} P_1 \text{ choices}$$

To find a mixed Nash Equilibria (MNE), compute vectors

$$y^* := [y_1^* \;\; 1 - y_1^*]', \qquad y_1^* \in [0, 1]$$
$$z^* := [z_1^* \;\; 1 - z_1^*]', \qquad z_1^* \in [0, 1]$$

for which

$$y^{*\prime} A z^* = y_1^*(1 - 6z_1^*) + 4z_1^* - 1 \leq y_1(1 - 6z_1^*) + 4z_1^* - 1, \quad \forall y_1 \in [0, 1]$$
$$y^{*\prime} B z^* = z_1^*(2 - 6y_1^*) + 5y_1^* - 2 \leq z_1(2 - 6y_1^*) + 5y_1^* - 2, \quad \forall z_1 \in [0, 1]$$

true if RHS of: eq. 1 independent of $y_1$; eq. 2 independent of $z_1$

# Completely Mixed Nash Equilibria

In particular, by making

$$\begin{cases} 1 - 6z_1^* = 0 \\ 2 - 6y_1^* = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} z_1^* = \frac{1}{6} \\ y_1^* = \frac{1}{3} \end{cases}$$

This leads to the following MNE

$$(y^*, z^*) = \underbrace{\left( \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}', \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix}' \right)}_{\substack{P_1 \text{ (husband) goes to football 66\% of times} \\ P_2 \text{ (wife) goes to football 83\% of times}}}$$

and outcomes

$$(y^{*\prime} A z^*, y^{*\prime} B z^*) = (4z_1^* - 1, 5y_1^* - 2) = \left( -\frac{1}{3}, -\frac{1}{3} \right)$$

# Completely Mixed Nash Equilibria

This particular NE has the very special property that

$$y^{*\prime} A z^* = y' A z^*, \quad \forall y \in \mathcal{Y} \qquad\qquad y^{*\prime} B z^* = y^{*\prime} B z, \quad \forall z \in \mathcal{Z}$$

then it is also a NE for a bimatrix game defined by the matrices $(-A, -B)$, i.e., exactly opposite objectives by both players.

# Completely Mixed Nash Equilibria

**Definition 10.1** (Completely Mixed Nash Equilibria (MNE)).

A MNE $(y^*, z^*)$ is **completely mixed** or an **inner-point equilibria** if all probabilities are strictly positive, i.e., $y_i^*, z_j^* \in (0, 1), \forall i, j$.

**Lemma 10.1** (Completely mixed NE).

If $(y^*, z^*)$ is a completely MNE with outcomes $(p^*, q^*)$ for a bimatrix game defined by the matrices $(A, B)$, then

$$Az^* = p^* \mathbf{1}_{m\times 1}, \qquad B' y^* = q^* \mathbf{1}_{n\times 1},$$

Consequently, $(y^*, z^*)$ is also a MNE for the three bimatrix games defined by $(-A, -B)$, $(A, -B)$ and $(-A, B)$.

# Completely Mixed Nash Equilibria

**Proof of Lemma 10.1.**

Assuming that $(y^*, z^*)$ is a completely MNE for the game defined by $(A, B)$, we have that

$$y^{*\prime} A z^* = \min_y y' A z^* = \min_y \sum_i y_i \underbrace{(Az^*)i}_{i\text{th row of } Az^*}$$

if row $i$ of $Az^*$ was strictly larger than any of the others, then the minimum would be achieved with $y_i = 0$ and the NE would not be completely mixed. To have a completely MNE, we need all the rows of $Az^*$ exactly equal to each other:

$$Az^* = p^* \mathbf{1}_{m \times 1}$$

for some scalar $p^*$, which means that

$$y^{*\prime} A z^* = y' A z^* = p^*, \qquad \forall y, y^* \in \mathcal{Y}$$

# Completely Mixed Nash Equilibria

Similarly, since none of the $z_j = 0$, then all columns of $y^{*\prime}B$ (the rows of $B'y^*$) must be equal to some constant $q^*$ and then

$$y^{*\prime}Bz^* = y^{*\prime}Bz = q^*, \qquad \forall z, z^* \in \mathcal{Z}$$

Therefore, we conclude $(p^*, q^*)$ is indeed the Nash outcome of the game and that $y^*, z^*$ is also a MNE for the three bimatrix games defined by $(-A, -B)$, $(A, -B)$ and $(-A, B)$.

**Corollary 10.1.** If $(y^*, z^*)$ is a completely mixed SPE for the zero-sum matrix game defined by $A$ with mixed value $p^*$, then

$$Az^* = p^*\mathbf{1}_{m\times 1} \qquad A'y^* = p^*\mathbf{1}_{n\times 1}$$

for some scalar $p^*$. Consequently, $(y^*, z^*)$ is also a mixed SPE for the zero-sum matrix game $-A$, and a MNE for the two (non-zero-sum) bimatrix games $(A, A)$ and $(-A, -A)$.

# Computation of Completely Mixed NE

# Computation of Completely Mixed Nash Equilibria

Simple: all the equilibria must satisfy (see **Lemma 10.1**)

$$B'y^* = q^*\mathbf{1}_{n\times 1} \qquad \mathbf{1}'y^* = 1 \qquad Az^* = p^*\mathbf{1}_{m\times 1} \qquad \mathbf{1}'z^* = 1$$

a linear system, with $n + m + 2$ equations and unknowns:

- $m$ entries of $y^*$, $n$ entries of $z^*$, and two scalars $p^*$ and $q^*$.

After solving, verify that the resulting $y^*$ and $z^*$ do have non-zero entries so that they belong to the sets $\mathcal{Y}$ and $\mathcal{Z}$

- if they do, we conclude that we have found a NE.

**Lemma 10.2.** Suppose that the vectors $(y^*, z^*)$ satisfy the previous conditions. If all entries of $y^*$ and $z^*$ are non-negative, then $(y^*, z^*)$ is a mixed NE.

# Computation of Completely Mixed Nash Equilibria

**Example 10.1** (Battle of the sexes BoS).

For the BoS game introduced before, the conditions become

$$Az^* = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} z_1^* \\ 1 - z_1^* \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix} \qquad B'y^* = \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} y_1^* \\ 1 - y_1^* \end{bmatrix} = \begin{bmatrix} q^* \\ q^* \end{bmatrix}$$

which is equivalent to

$$p^* = -2z_1^*, \quad 4z_1^* = 1 + p^*, \quad q^* = -y_1^*, \quad 5y_1^* = 2 + q^*$$

$$\Rightarrow z_1^* = \frac{1}{6}, \quad y_1^* = \frac{1}{3}, \quad p^* = q^* = \frac{1}{3}$$

as we had previously concluded.

However, we now know that this is the unique completely MNE for this game.

# Computation of Completely Mixed Nash Equilibria

For the BoS game in Example 9.3, the conditions become

$$Az^* = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} z_1^* \\ 1 - z_1^* \end{bmatrix} = \begin{bmatrix} p^* \\ p^* \end{bmatrix} \qquad B'y^* = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1^* \\ 1 - y_1^* \end{bmatrix} = \begin{bmatrix} q^* \\ q^* \end{bmatrix}$$

which is equivalent to

$$-3z_1 + 1 = p^*, \quad z_1^* - 1 = p^*, \quad -4y_1^* + 3 = q^*, \quad 4y_1^* - 2 = q^*$$

$$\Rightarrow z_1^* = \frac{1}{2}, \quad y_1^* = \frac{5}{8}, \quad p^* = -\frac{1}{2}, \quad q^* = \frac{1}{2}$$

**Note:** the completely MNE is not admissible: it is **strictly worse** for both players than the pure NE that we found before.

- $(1, 1)$ was a pure NE with outcomes $(-2, -1)$
- $(2, 2)$ was a pure NE with outcomes $(-1, -2)$

# Numerical Computation of MNE

# Numerical Computation of Mixed Nash Equilibria

A systematic numeric procedure to find MNE for a bimatrix game defined by two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ expressing the outcomes of players $P_1$ and $P_2$.

$P_1$ selects a row of $A/B$ and $P_2$ selects a column of $A/B$.

MNE can be found by solving a **quadratic program**

# Numerical Computation of Mixed Nash Equilibria

**Theorem 10.1.** The pair of policies $(y^*, z^*) \in \mathcal{Y} \times \mathcal{Z}$ is a MNE with outcome $(p^*, q^*)$ if and only if the tuple $(y^*, z^*, p^*, q^*)$ is a (global) solution to the following minimization:

$$\underbrace{\begin{array}{ll} \text{minimize} & y'(A+B)z - p - q \\ \text{subject to} & \begin{array}{rcl} Az & \dot{\geq} & p\mathbf{1} \\ B'y & \dot{\geq} & q\mathbf{1} \\ \left.\begin{array}{rcl} y & \dot{\geq} & 0 \\ \mathbf{1}y & = & 1 \end{array}\right\} & (y \in \mathcal{Y}) \\ \left.\begin{array}{rcl} z & \dot{\geq} & 0 \\ \mathbf{1}z & = & 1 \end{array}\right\} & (z \in \mathcal{Z}) \end{array} \end{array}}_{\text{optimization over } m+n+2 \text{ parameters } (y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_n, p, q)}$$

This minimization always has a global minima at zero.

# Numerical Computation of Mixed Nash Equilibria

For zero-sum games, $A = -B$ and minimization is a linear program that finds both policies in one shot

**Observation:** efficiency is reduced!

- solving two small problems is better than solving a large one.

**Attention!** Unless $A = -B$ (a zero-sum game), the quadratic criteria is indefinite because we can select $z$ to be any vector for which $(A + B)z \neq 0$ and then obtain

A positive value with

$$y = (A + B)z, \quad p = q = 0 \quad \Rightarrow \quad y'(A + B)'(A + B)z - p - q = ||(A + B)z||^2 > 0$$

and a negative value with

$$y = -(A + B)z, \quad p = q = 0 \Rightarrow y'(A + B)'(A + B)z - p - q = -||(A + B)z||^2 < 0$$

# Numerical Computation of Mixed Nash Equilibria

Here, the quadratic criteria in the minimization is not convex: numerical solvers can get caught in local minima.

- **Solution:** verify that solver found a **global minimum**.

Easily done: we know that the global minimum is **zero**.

If solver gets caught in a local minimum

- restart it at a different (typically random) initial point.

# Numerical Computation of Mixed Nash Equilibria

**MATLAB® Hint 3** (Quadratic programs).

**MATLAB®**'s Optimization Toolbox command

```
[x,val] = quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0)
```

numerically solves quadratic programs of the form

$$\begin{array}{ll} \text{minimum} & \frac{1}{2}\texttt{x'Hx + c'x} \\ \text{subject to} & \texttt{Ain x} \dot{\leq} \texttt{bin} \\ & \texttt{Aeq x} = \texttt{beq} \\ & \texttt{low} \dot{\leq} \texttt{x} \dot{\leq} \texttt{high} \end{array}$$

and returns the value `val` of the minimum and a vector `x` that achieves the minimum.

To avoid the corresponding inequality constraints, the vector

- `low`: can have some or all entries equal to `-Inf`
- `high`: can have some or all entries equal to `+Inf`

# Numerical Computation of Mixed Nash Equilibria

Vector `x0`: (optional) starting point for the optimization

- important when `H` is indefinite since in this case the minimization is not convex and may have local minima

**MATLAB**® command `optimset`

- used to set optimization options for `quadprog`
- allows the selection of the optimization algorithm, which for the computation of NE, must support non-convex problems.

The following **MATLAB**® code can be used to find a MNE to the bimatrix game defined by $A$ and $B$, starting from a random initial condition `x0`.

# Numerical Computation of Mixed Nash Equilibria

```
[m,n] = size(A);
x0 = rand(n+m+2,1);

% y'( A + b )z - p - q
H = [zeros(m,m), A+B, zeros(m,2); A'+B', zeros(n,n+2); zeros(2, m+n+2)];
c = [zeros(m+n,1); -1; -1];

% Az >= p & B' y >= q
Ain = [zeros(m,m), -A, ones(m,1), zeros(m,1); -B', zeros(n,n+1), ones(n,1)];
bin = zeros(m+n,1);

% sum(y) = sum(z)=1
Aeq = [ones(1,m), zeros(1,n+2); zeros(1,m), ones(1,n),0 ,0];
beq = [1;1];

% y_i , z_i in [0,1]
low = [zeros(n+m,1); -inf;-inf];
high = [ones(n+m,1);+inf;+inf];

% solve quadratic program
options = optimset('TolFun',1e-8,'TolX',1e-8,'TolPCG',1e-8,'Algorithm','active-set');

[x,val,exitflag ] = quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0,options)

y = x(1:m)
z = x(m+1:m+n)
p = x(m+n+1)
q = x(m+n+2)
```

# Numerical Computation of Mixed Nash Equilibria

**Proof of Theorem 10.1.** Justify two statements separately:

1. a MNE is always a global minimum
2. any global minimum is a MNE

For **1**, assume $(y^*, z^*)$ is a MNE with outcome $(p^*, q^*)$ and show that $(y^*, z^*, p^*, q^*)$ is a global minimum. We need to show:

**1.** Point $(y^*, z^*, p^*, q^*)$ satisfies all the constraints in the minimization. Indeed, since

$$p^* = y^{*\prime} A z^* \leq y' A z^*, \quad \forall y \in \mathcal{Y}$$

in particular for every integer $i \in \{1, 2, \dots, m\}$ if we pick

$$y = \underbrace{[0 \ \ \cdots \ \ 0 \ \ 1 \ \ 0 \ \ \cdots \ \ 0]'}_{1 \text{ at the } i\text{th position}}$$

# Numerical Computation of Mixed Nash Equilibria

we conclude that

$$p^* \leq \underbrace{(Az^*)i}_{i\text{th entry of } Az^*}$$

and therefore $p^* \mathbf{1} \dot{\leq} Az^*$. On the other hand, since

$$q^* = y^{*\prime} B z^* \leq y^{*\prime} B z, \quad \forall z \in \mathcal{Z}$$

we also conclude that $q^* \mathbf{1}' \dot{\leq} y^* B$. This entry-wise inequality between row vectors can be converted into an entry-wise inequality between column vectors by transposition: $q^* \mathbf{1} \dot{\leq} B' y^*$.

Remaining constraints on $y^*$ and $z^*$ hold since $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$.

# Numerical Computation of Mixed Nash Equilibria

**2.** $(y^*, z^*, p^*, q^*)$ achieves the global minimum, which is zero. Note that since $p^* = y^{*\prime} A z^*$ and $q^* = y^{*\prime} B z^*$ we have

$$y^{*\prime}(A+B)z^* - p^* - q^* = 0$$

It remains to show that no other vectors $y$ and $z$ that satisfy the constraints can lead to a value for the criteria lower than zero:

$$\begin{cases} Az \geq p\mathbf{1} \\ B'y \geq q\mathbf{1} \end{cases} \Rightarrow \begin{cases} y'Az \geq p \\ z'B'y \geq q \end{cases} \Rightarrow y'(A+B)z - p - q \geq 0$$

To prove the converse statement assume $(y^*, z^*, p^*, q^*)$ is a global minimum, and show that $(y^*, z^*)$ must be a MNE with outcome $(p^*, q^*)$.

From **Theorem 9.1**: there is at least one MNE and we have seen above that this leads to a global minimum equal to zero.

# Numerical Computation of Mixed Nash Equilibria

Therefore any global minimum $(y^*, z^*, p^*, q^*)$ must satisfy:

$$y^{*\prime}(A+B)z^* - p^* - q^* = 0 \qquad \begin{array}{l} Az^* \dot{\geq} p^* \mathbf{1}, \quad z^* \in \mathcal{Z} \\ B'y^* \dot{\geq} q^* \mathbf{1}, \quad y^* \in \mathcal{Y} \end{array}$$

From RHS eqs., we conclude that

$$y'Az^* \geq p^*, \quad \forall y \in \mathcal{Y} \qquad z'B'y^* \geq q^*, \quad \forall z \in \mathcal{Z}$$

The proof is completed as soon as we show that

$$y^*Az^* = p^*, \qquad z^{*\prime}B'y^* = q^*$$

To achieve this, set $y = y^*$ and $z = z^*$, which leads to

$$y'Az^* - p^* \geq 0, \qquad z^{*\prime}B'y^* - q^* \geq 0$$

However, because of LHS eq., the two numbers in the LHS of these ineq. must add up to zero, so they must be equal to zero. $y^*Az^* = p^*$ and $z^{*\prime}B'y^* = q^*$ hold, and this completes the proof.

# Practice Exercises

# Practice Exercises

**10.1.** Use **MATLAB®** to compute NE policies for

1. battle of the sexes with

$$A = \underbrace{\begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices} \qquad B = \underbrace{\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices}$$

2. prisoners' dilemma with

$$A = \underbrace{\begin{bmatrix} 2 & 30 \\ 0 & 8 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices} \qquad B = \underbrace{\begin{bmatrix} 2 & 0 \\ 30 & 8 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} P_1 \text{ choices}$$

**Solution to Exercise 10.1.** The following **MATLAB®** code can be used to solve both problems.

# Practice Exercises

```
[m,n] = size(A);
x0 = rand(n+m+2,1);

% y'( A + b )z -p - q
H = [zeros(m,m), A+B, zeros(m,2); A'+B', zeros(n,n+2); zeros(2, m+n+2)];
c = [zeros(m+n,1); -1; -1];

% Az >= p & B'y >= q
Ain = [zeros(m,m), -A, ones(m,1), zeros(m,1); -B', zeros(n,n+1), ones(n,1)];
bin = zeros(m+n,1);

% sum(y) = sum(z)=1
Aeq = [ones(1,m), zeros(1,n+2); zeros(1,m), ones(1,n),0 ,0];
beq = [1;1];

% y_i , z_i in [0,1]
low = [zeros(n+m,1); -inf; -inf];
high = [ones(n+m,1);+inf; +inf];

% solve quadratic program
options = optimset('TolFun',1e-8,'TolX',1e-8,'TolPCG',1e-8,'Algorithm','active-set');

[x,val,exitflag ] = quadprog(H,c,Ain,bin,Aeq,beq,low,high,x0,options)

y = x(1:m)
z = x(m+1:m+n)
p = x(m+n+1)
q = x(m+n+2)
```

# Practice Exercises

1. For the BoS game, the code should be preceded by

```
A = [-2,0; 3,-1];                    B = [-1,3; 0,-2];
```

and we obtain the following global minimum

```
y = 0.3333        p = -0.3333
    0.6667
z = 0.1667        q = -0.3333
    0.8333
```

2. For the prisoners' dilemma, the code should be preceded by

```
A =[2,30; 0,8];                    B =[2,0; 30 ,8];
```

and we obtain the following global minimum

```
y = 0.0000        p = 8.0000
    1.0000
z = 0.0000        q = 8.0000
    1.0000
```

15% of the time did we get a global minima.
The remaining 85% of the cases, we obtained local minima.

End of Lecture

**10 - Computation of Nash Equilibria for Bimatrix Games**

Questions?