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Games: Theory and Applications

Lecture 11 - N-Player Games

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N-Player Games

Games with N-**players** $P_1, P_2, \ldots, P_N$, allowed to select policies within **action spaces** $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$. When

$$\begin{cases} P_1 \text{ uses policy } \gamma_1 \in \Gamma_1 \\ P_2 \text{ uses policy } \gamma_2 \in \Gamma_2 \\ \vdots \\ P_N \text{ uses policy } \gamma_N \in \Gamma_N \end{cases}$$

the **outcome of the game** for player P_i is denoted by

$$J_i(\gamma_1, \gamma_2, \ldots, \gamma_N)$$

Each P_i wants to **minimize** their own outcome, and does not care about the outcome of the other players.

N-Player Games

To avoid writing all the policies, separate the dependence of J_i on γ_i and on the remaining policies γ_{-i} and write

$$J_i(\gamma_i, \gamma_{-i})$$

with the abbreviation to denote a list of all but the ith policy

$$\gamma_{-i} \equiv (\gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_N)$$

Terminology also applies to action spaces, as in

$$\gamma_{-i} \in \Gamma_{-i}$$

which is meant to be a short-hand notation for

$$\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2, \ldots, \gamma_{i-1} \in \Gamma_{i-1}, \gamma_{i+1} \in \Gamma_{i+1}, \ldots \gamma_N \in \Gamma_N$$

Security Levels and Policies

Security policies for N-player games:

Finding the policy that guarantees the least possible cost, assuming the worse possible choice by the other players.

Definition 11.1 (Security policy).

Security level for P_i, $i \in \{1, 2, \dots, N\}$ is defined by

$$\bar{V}(J_i) := \underbrace{\inf_{\gamma_i \in \Gamma_i}}_{\substack{\text{minimize cost assuming} \\ \text{worst choice by } P_i}} \underbrace{\sup_{\gamma_{-i} \in \Gamma_{-i}}}_{\substack{\text{worst choice by all} \\ \text{other players } P_{-i} \\ \text{from } P_i\text{'s perspective}}} I_i(\gamma_i, \gamma_{-i})$$

Security Levels and Policies

Security policy for P_i

Any policy $\gamma_i^* \in \Gamma_i$ for which the infimum is achieved, i.e.,

$$\bar{V}(J_i) := \inf_{\gamma_i \in \Gamma_i} \sup_{\gamma_{-i} \in \Gamma_{-i}} J_i(\gamma_i, \gamma_{-i}) = \underbrace{\sup_{\gamma_{-i} \in \Gamma_{-i}} J_i(\gamma_i^*, \gamma_{-i}^*)}_{\gamma_i^* \text{ achieves the infimum}}$$

Security policies may not exist because the infimum may not be achieved by a policy in Γ_i.

An N-tuple of policies $(\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is said to be **minimax** if each γ_i is a security policy for P_i.

Nash Equilibria

Definition 11.2 (Nash equilibrium). An N-tuple of policies

$$\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$$

is a NE if

$$J_i(\gamma^*) = J_i(\gamma_i^*, \gamma_{-i}^*) \leq J_i(\gamma_i, \gamma_{-i}^*), \quad \forall \gamma_i \in \Gamma_i, \quad i \in \{1, 2, \ldots, N\}$$

and the N-tuple $(J_1(\gamma^*), J_2(\gamma^*), \ldots, J_N(\gamma^*))$ is called the **Nash outcome of the game**.

The NE is **admissible** if there is no **better** NE in the sense that there is no other $\bar{\gamma}^* := (\bar{\gamma}_1^*, \bar{\gamma}_2^*, \ldots, \bar{\gamma}_N^*) \in \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_N$ such that

$$J_i(\bar{\gamma}^*) \leq J_i(\gamma^*), \quad \forall i \in \{1, 2, \ldots, N\}$$

with a strict inequality for at least one player.

Pure N-Player Games in Normal Form

Pure N-Player Games in Normal Form

Played by N players $P_1, P_2, \ldots, P_N$, each selecting policies from finite **action spaces**:

$$P_i \text{ has available } m_i \text{ actions: } \Gamma_i := \{1, 2, \ldots, m_i\}$$

Outcomes for P_i's are quantified by N tensors $A^1, A^2, \ldots, A^N$, each N-dimensional with dimensions $m_1, m_2, \ldots, m_N$. When

$$\begin{cases} P_1 \text{ selects action } k_1 \in \Gamma_1 := \{1, 2, \ldots, m_1\} \\ P_2 \text{ selects action } k_2 \in \Gamma_2 := \{1, 2, \ldots, m_2\} \\ \vdots \\ P_N \text{ selects action } k_N \in \Gamma_N := \{1, 2, \ldots, m_N\} \end{cases}$$

the **outcome** for P_i is obtained from the appropriate entry $a^i_{k_1 k_2 \cdots k_N}$ of the tensor A^i

- all players want to **minimize** their respective outcomes.

Pure N-Player Games in Normal Form

Testing if a particular N-tuple of pure policies $(k_1^*, k_2^*, \ldots, k_N^*)$ is a NE is straightforward. Just check if

$$a^i_{k_i^* k_{-i}^*} \leq a^i_{k_i k_{-i}^*}, \qquad \forall k_i \in \{1, 2, \ldots, m_i\}, \quad \forall i \in \{1, 2, \ldots, N\}$$

Finding a NE in pure policies is computationally difficult

- need to check all possible N-tuples, which are as many as

$$m_1 \times m_2 \times \cdots \times m_N$$

Tensor: a multi-dimensional array that generalizes the concept of matrix for dimensions higher than two.

Mixed Policies for N-Player Games in Normal Form

Mixed Policies for N-Player Games in Normal Form

A **mixed policy for player** P_i is a set of numbers

$$y^i := (y^i_1, y^i_2, \ldots, y^i_{m_i}), \qquad \sum_{k=1}^{m_i} y^i_k = 1 \qquad y^i_k \geq 0, \quad \forall k \in \{1, 2, \ldots, m_i\}$$

y^i_k: probability that P_i uses to select action $k \in \{1, 2, \ldots, m_i\}$.

Each mixed policy y_i is an element of the action space $\mathcal{Y}^i$, consisting of probability distributions over m_i actions.

Random selections by P_i's are statistically independently

- each P_i tries to minimize their own **expected outcome**:

$$J_i = \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \cdots \sum_{k_N=1}^{m_N} \underbrace{y^1_{k_1} y^2_{k_2} \cdots y^N_{k_N}}_{\text{probability that } P_1 \text{ selects } k_1 \text{ and } P_2 \text{ selects } k_2 \text{ and } \ldots} \qquad \underbrace{a_{k_1 k_2 \cdots k_N}}_{\text{outcome when } P_1 \text{ selects } k_1 \text{ and } P_2 \text{ selects } k_2 \text{ and } \ldots}$$

Mixed Policies for N-Player Games in Normal Form

Definition 11.3 (Mixed Nash equilibrium).

An N-tuple of policies $(y^{1*}, y^{2*}, \dots, y^{N*}) \in \mathcal{Y}^1 \times \mathcal{Y}^2 \times \cdots \times \mathcal{Y}^N$ is a **mixed Nash equilibrium** (MNE) if

$$\sum_{k_1}\sum_{k_2}\cdots\sum_{k_N} \boxed{y^{1*}_{k_1}} y^{2*}_{k_2}\cdots y^{N*}_{k_N} a^1_{k_1 k_2 \cdots k_N} \leq \sum_{k_1}\sum_{k_2}\cdots\sum_{k_N} \boxed{y^{1}_{k_1}} y^{2*}_{k_2}\cdots y^{N*}_{k_N} a^1_{k_1 k_2 \cdots k_N}$$

$$\sum_{k_1}\sum_{k_2}\cdots\sum_{k_N} y^{1*}_{k_1} \boxed{y^{2*}_{k_2}} \cdots y^{N*}_{k_N} a^1_{k_1 k_2 \cdots k_N} \leq \sum_{k_1}\sum_{k_2}\cdots\sum_{k_N} y^{1*}_{k_1} \boxed{y^{2}_{k_2}} y^{3*}_{k_3}\cdots y^{N*}_{k_N} a^2_{k_1 k_2 \cdots k_N}$$

$\dots$

or equivalently in a more compressed form

$$\sum_{k_1 k_2 \cdots k_N} \boxed{y^{i*}_{k_i}} \left(\prod_{j\neq i} y^{j*}_{k_j}\right) a^i_{k_1 k_2 \cdots k_N} \leq \sum_{k_1 k_2 \cdots k_N} \boxed{y^{i}_{k_i}} \left(\prod_{j\neq i} y^{j*}_{k_j}\right) a^i_{k_1 k_2 \cdots k_N},$$

$\forall i \in \{1, 2, \dots, N\}$

Mixed Policies for N-Player Games in Normal Form

(As in bimatrix games) The introduction of mixed policies enlarges the action spaces for both players to the point that NE now always exist.

Theorem 11.1 (Nash).

Every N-player game in normal form has at least one mixed Nash Equilibrium.

Completely Mixed Policies

Completely Mixed Policies

Computing NE for N-player games in normal form is not easy

- simpler for games that admit completely mixed equilibria

Definition 11.4 (completely mixed Nash equilibria (MNE))

A MNE $(y^{1*}, y^{2*}, \ldots, y^{N*})$ is **completely mixed** or an **inner-point equilibrium** if all probabilities are strictly positive, i.e.,

$$y^{1*} \dot{>} 0, \quad y^{2*} \dot{>} 0, \cdots, y^{N*} \dot{>} 0,$$

All completely MNE can be found by solving an algebraic multi-linear system of equations.

Completely Mixed Policies

Lemma 11.1 (completely mixed Nash equilibria).
If $(y^{1*}, y^{2*}, \ldots, y^{N*})$ is a completely MNE with outcomes $(p^{1*}, p^{2*}, \ldots, p^{N*})$ then

$$\sum_{k_{-i}} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N} = p^{i*}, \qquad \forall i \in \{1, 2, \ldots, N\}$$

Conversely, any solution $(y^{1*}, \ldots, y^{N*}), (p^{1*}, \ldots, p^{N*})$ for which

$$\sum_{k_i=1}^{m_i} y^{i*}_{k_i} = 1, \qquad y^{i*} \dot{\geq} 0, \quad \forall i \in \{1, 2, \ldots, N\}$$

corresponds to a MNE $(y^{1*}, y^{2*}, \ldots, y^{N*})$ with outcomes $(p^{1*}, p^{2*}, \ldots, p^{N*})$ for the original game, and for any similar game in which some/all players want to maximize instead of minimize their outcomes.

Completely Mixed Policies

Proof of Lemma 11.1.

Assuming $(y^{1*}, y^{2*}, \ldots, y^{N*})$ is a completely MNE, we have

$$\sum_{k_1 k_2 \cdots k_N} y^{i*}_{k_i} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N} = \min_{y^i} \sum_{k_1 k_2 \cdots k_N} y^{i}_{k_i} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N}$$

$$= \min_{y^i} \sum_{k_i} y^{i}_{k_i} \sum_{k_{-i}} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N}$$

If one of the $\sum_{k_{-i}} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N}$ was strictly larger than any of the remaining ones, then the minimum would be achieved with $y_i = 0$ and the NE would not be completely mixed. Therefore to have a completely MNE, we must have $\sum_{k_{-i}} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N} = p^{i*}$

Completely Mixed Policies

Conversely, if $(y^{1*}, y^{2*}, \ldots, y^{N*})$ and $(p^{1*}, p^{2*}, \ldots, p^{N*})$ satisfy the two conditions in **Lemma 11.1**, , then

$$\sum_{k_1 k_2 \cdots k_N} y^{i*}_{k_i} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N} = \sum_{k_1 k_2 \cdots k_N} y^{i}_{k_i} \left(\prod_{j \neq i} y^{j*}_{k_j} \right) a^i_{k_1 k_2 \cdots k_N}$$
$$= \min_{y^i} \sum_{k_i} y^i_{k_i} p^{i*} = p^{i*}, \quad \forall y^i \in \mathcal{Y}^i$$

which shows that $(y^{1*}, y^{2*}, \ldots, y^{N*})$ is a MNE with outcome $(p^{1*}, p^{2*}, \ldots, p^{N*})$.

In fact, $(y^{1*}, y^{2*}, \ldots, y^{N*})$ is also a MNE for a different game in which some/all P_i's want to maximize instead of minimize the outcome.

End of Lecture

11 - N-Player Games

Questions?