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Games: Theory and Applications

Lecture 12 - Potential Games

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Games with N-**players** $P_1, P_2, \ldots, P_N$, who select policies from **action spaces** $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$.

If P_i uses policy $\gamma_i \in \Gamma_i$ the **outcome of the game** for P_i is

$$J_i(\gamma_1, \gamma_2, \ldots, \gamma_N) \in \Gamma := \Gamma_1 \times \Gamma_2 \times \cdots \Gamma_N$$

Each P_i wants to **minimize** their own outcome.

A game is of **identical interests** (II) if all players have the same outcome, i.e., if there exists a function $\phi(\gamma)$ such that

$$J_i(\gamma) = \phi(\gamma), \qquad \forall \gamma \in \Gamma, \quad i \in \{1, 2, \ldots, N\}$$

Identical Interests Games

Notion of NE for II games is related to the notion of minimum

A given $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is a **global minimum** of ϕ if

$$\phi(\gamma^*) \leq \phi(\gamma), \qquad \forall \gamma \in \Gamma$$

and is a **directionally-local minimum** of ϕ if

$$\phi(\gamma_i^*, \gamma_{-i}^*) \leq (\gamma_i, \gamma_{-i}^*), \qquad \forall \gamma_i \in \Gamma_i, \quad i \in \{1, 2, \ldots, N\}$$

Note. Every global minimum is also a directionally-local minimum.

Identical Interests Games

Proposition 12.1.

Consider an II game with outcomes given by $J_i(\gamma) = \phi(\gamma)$

1.- An N-tuple of policies $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is a NE if and only if γ^* is a directionally-local minimum of ϕ.

2. Assume the **potential function** ϕ has at least one global minimum. An N-tuple of policies $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is an admissible NE if and only if γ^* is a global minimum to the function ϕ in $J_i(\gamma) = \phi(\gamma)$.
Moreover, all admissible NE have the same value for all players: the global minimum of ϕ.

Identical Interests Games

Proof of Proposition 12.1.

Statement 1: a consequence of the fact that for an II game with outcomes given by $J_i(\gamma) = \phi(\gamma)$ the definition of a NE is precisely the condition $\phi(\gamma_i^*, \gamma_{-i}^*) \leq (\gamma_i, \gamma_{-i}^*)$, $\forall \gamma_i \in \Gamma_i$.

Statement 2: suppose $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is a global minimum, then

$$J_i(\gamma_i^*, \gamma_{-i}^*) := \phi(\gamma_i^*, \gamma_{-i}^*) \leq \phi(\gamma_i, \gamma_{-i}^*) =: J_i(\gamma_i, \gamma_{-i}^*) \qquad \forall \gamma_i \in \Gamma_i, \quad i \in \{1, 2, \ldots, N\}$$

which shows that γ^* is a NE.

Since γ^* is a global minimum of ϕ, it must be admissible since no other policies could lead to a smaller value of the outcomes.

Identical Interests Games

Conversely, suppose that $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \Gamma$ is not a global minimum

- now $\bar{\gamma}^* \neq \gamma^*$ is a global minimum.

From previous statements, we conclude $\bar{\gamma}^*$ must be a NE that leads to outcomes that are **better** than those of γ^* for all $P_i's$

- meaning: even if γ^* is a NE, it cannot be admissible.

II games are attractive because all global minima of the common outcome ϕ of all players (called the social optima) are admissible NE.

Identical Interests Games

Keep in mind the following two facts:

1.- Even though all admissible NE are global minima, there may be (non-admissible) NE that are not global minima.

If players choose non-admissible NE policies, they could be playing at an equilibrium that is not a global minimum.

Example: the pure bimatrix game defined by

$$A = B = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Game has a single pure admissible NE $(1, 1)$ that corresponds to the global minimum 0. Another non-admissible NE $(2, 2)$ corresponds to the outcome 1 for both players.

Identical Interests Games

2.- There may be problems if multiple global minima exist, because even though all admissible NE have the same outcome for all players, they **may not enjoy the order interchangeability property**.

Example: the pure bimatrix game defined by

$$A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Game has two pure admissible NE $(1,1)$ and $(2,2)$, both corresponding to the global minimum 0.

However, $(1,2)$ and $(2,1)$ are not global minima.

Potential Games

Potential Games

A game is an **(exact) potential game** if there exists a function $\phi(\gamma_1, \gamma_2, \ldots, \gamma_N)$ such that

$$J_i(\gamma_i, \gamma_{-i}) - J_i(\bar{\gamma}_i, \gamma_{-i}) = \phi(\gamma_i, \gamma_{-i}) - \phi(\bar{\gamma}_i, \gamma_{-i}), \quad \forall \gamma_i, \bar{\gamma}_i \in \Gamma_i, \gamma_{-i} \in \Gamma_{-i} \quad i \in \{1, 2, \ldots, N\}$$

and ϕ is called an **(exact) potential** for the game.

In words: If a player P_i unilateral deviates from γ_i to $\bar{\gamma}_i$, the change in its outcome is exactly equal to the change in the potential, which is common to all players.

Potential Games

A game is an **ordinal potential game** if there exists a function $\phi(\gamma_1, \gamma_2, \ldots, \gamma_N)$ such that

$$J_i(\gamma_i, \gamma_{-i}) - J_i(\bar{\gamma}_i, \gamma_{-i}) > 0 \Leftrightarrow \phi(\gamma_i, \gamma_{-i}) - \phi(\bar{\gamma}_i, \gamma_{-i}) > 0, \quad \forall \gamma_i, \bar{\gamma}_i \in \Gamma_i, \gamma_{-i} \in \Gamma_{-i}, \quad i \in \{1, 2, \ldots, N\}$$

and ϕ is called an **ordinal potential** for the game.

In words: If a player P_i unilateral deviates from γ_i to $\bar{\gamma}_i$, the sign of the change in its outcome is equal to the sign of the change in the potential, which is common to all players.

Potential Games

Exact or ordinal potentials of a game are not uniquely defined.

- if $\phi(\cdot)$ is an exact/ordinal potential then, for every constant c, $\phi(\cdot) + c$ is also an exact/ordinal potential for the same game.

For **exact potential games**, while the potential is not unique, all potentials differ only by an additive constant

- if ϕ and $\bar{\phi}$ are both potentials for the same exact potential game, there must exist a constant c such that

$$\phi(\gamma) = \bar{\phi} + c \quad \forall \gamma \in \Gamma$$

Minima vs. Nash Equilibria in Potential Games

Directionally-local minima of the potential ϕ are NE

Proposition 12.2. Consider an exact or ordinal potential game with (exact or ordinal) potential ϕ.

An N-tuple of policies $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is a NE if and only if γ^* is a directionally-local minimum of ϕ.

When the action spaces are finite, a global minimum always exists and therefore a directionally-local minimum also exists. In this case, potential games always have at least one NE.

Proof of Proposition 12.2. Assuming $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is a directionally-local minimum of ϕ, we have that

$$\phi(\gamma_i^*, \gamma_{-i}^*) - \phi(\gamma_i, \gamma_{-i}^*) \leq 0, \quad \forall \gamma_i \in \Gamma_i, \quad i \in \{1, 2, \ldots, N\}$$

Minima vs. Nash Equilibria in Potential Games

But, both for exact and ordinal games, we also have that

$$J_i(\gamma_i^*, \gamma_{-i}^*) - J_i(\gamma_i, \gamma_{-i}^*) \leq 0, \qquad \forall \gamma_i \in \Gamma_i, \quad i \in \{1, 2, \ldots, N\}$$

which shows that γ^* is indeed a NE.

Conversely, if $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is a NE, then the previous eq. holds, which for both exact and ordinal games, implies that $\phi(\gamma_i^*, \gamma_{-i}^*) - \phi(\gamma_i, \gamma_{-i}^*) \leq 0$ also holds.

From here we conclude that $\gamma^* := (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*)$ is a directionally-local minimum of ϕ.

Minima vs. Nash Equilibria in Potential Games

Attention!

For potential games there is an equivalence between directionally-local minima and NE, but there is no match between global minima and admissible NE.

Consider a pure bimatrix game defined by the matrices

$$A = [a_{ij}]_{2\times 2} = \begin{bmatrix} \alpha_1 & \alpha_2 + 1 \\ \alpha_1 + 1 & \alpha_2 \end{bmatrix} \qquad B = [b_{ij}]_{2\times 2} = \begin{bmatrix} \beta_1 & \beta_1 + 1 \\ \beta_2 + 1 & \beta_2 \end{bmatrix}$$

This is an exact potential game with potential given by

$$\Phi = [\phi_{ij}]_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Minima vs. Nash Equilibria in Potential Games

$$\Phi = [\phi_{ij}]_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Potential Φ has two global minima. Then we have two NE:

- $(1, 1)$ with outcomes (α_1, β_1)
- $(2, 2)$ with outcomes (α_2, β_2)

Different values of the constants $\alpha_1, \alpha_2.\beta_1, \beta_2$ can make only one or both of these NE admissible.

This shows that global minima of the potential may not generate admissible NE.

Bimatrix Potential Games

A pure bimatrix game defined by the two $m \times n$ matrices

$$A = [a_{ij}]_{m\times n} \qquad\qquad B = [b_{ij}]_{m\times n}$$

is an (exact) potential game, if there exists a potential

$$\Phi = [\phi_{ij}]_{m\times n}$$

such that

$$\begin{aligned} a_{ij} - a_{\bar{i}j} &= \phi_{ij} - \phi_{\bar{i}j} \qquad \forall i, \bar{i} \in \{1,2,\ldots,m\},\;\; j \in \{1,2,\ldots,n\} \\ b_{ij} - b_{i\bar{j}} &= \phi_{ij} - \phi_{i\bar{j}} \qquad \forall i \in \{1,2,\ldots,m\},\;\; j,\bar{j} \in \{1,2,\ldots,n\} \end{aligned}$$

To verify whether or not such a potential exists, regard the previous eqs. as a linear system of equations with

$$\frac{m(m-1)}{2}n + \frac{n(n-1)}{2}m$$

equations and mn unknowns (the entries of Φ).

Bimatrix Potential Games

When these equations have a solution, we can conclude that we have an exact potential game in pure policies.

These equations also guarantee that the bimatrix game is an (exact) potential game in mixed policies

Proposition 12.3 (Potential bimatrix games).

A bimatrix game defined by the $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is an exact potential game in pure or mixed policies if and only if there exists an $m \times n$ matrix $\Phi = [\phi_{ij}]$ for which the conditions below hold:

$$a_{ij} - a_{\bar{i}j} = \phi_{ij} - \phi_{\bar{i}j} \qquad \forall i, \bar{i} \in \{1, 2, \dots, m\}, \quad j \in \{1, 2, \dots, n\}$$
$$b_{ij} - b_{i\bar{j}} = \phi_{ij} - \phi_{i\bar{j}} \qquad \forall i \in \{1, 2, \dots, m\}, \quad j, \bar{j} \in \{1, 2, \dots, n\}$$

Bimatrix Potential Games

Proof: a consequence of the fact that the conditions

$$a_{ij} - a_{\bar{i}j} = \phi_{ij} - \phi_{\bar{i}j} \qquad \forall i, \bar{i} \in \{1, 2, \ldots, m\}, \;\; j \in \{1, 2, \ldots, n\}$$
$$b_{ij} - b_{i\bar{j}} = \phi_{ij} - \phi_{i\bar{j}} \qquad \forall i \in \{1, 2, \ldots, m\}, \;\; j, \bar{j} \in \{1, 2, \ldots, n\}$$

correspond to the definition of an (exact) potential game.

For mixed policies, these conditions must be necessary

- pure policies are special cases of mixed policies.

Even in mixed policies, the equalities for **exact potential games** must hold for pure policies.

Bimatrix Potential Games

To prove that the conditions are also sufficient, we show that $y'\Phi z$ is a potential for the mixed bimatrix game.

Specifically, show that given arbitrary mixed policies

- $y, \bar{y}$ for P_1 and $z, \bar{z}$ for P_2

we have

$$y'Az - \bar{y}Az = y'\Phi z - \bar{y}\Phi z \qquad y'Bz - yB\bar{z} = y'\Phi z - y\Phi\bar{z}$$

To this effect, we start by expanding

$$y'Az - \bar{y}Az = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_i - \bar{y}_i)a_{ij}z_j \qquad y'Bz - yB\bar{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}y_i b_{ij}(z_j - \bar{z}_j)$$

Bimatrix Potential Games

Because of the conditions, we conclude that for every $\bar{i}$ and $\bar{j}$, these differences must equal

$$y'Az - \bar{y}Az = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_i - \bar{y}_i)(a_{\bar{i}j} + \phi_{ij} - \phi_{\bar{i}j})z_j$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}(y_i - \bar{y}_i)\phi_{ij}z_j + \left(\sum_{i=1}^{m}(y_i - \bar{y}_i)\right)\left(\sum_{j=1}^{n}(a_{\bar{i}j} - \phi_{\bar{i}j})z_j\right)$$

$$y'Bz - yB\bar{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}y_i(b_{i\bar{j}} + \phi_{ij} - \phi_{i\bar{j}})(z_j - \bar{z}_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}y_i\phi_{ij}(z_j - \bar{z}_j) + \left(\sum_{i=1}^{m}y_i(b_{i\bar{j}} - \phi_{i\bar{j}})\right)\left(\sum_{j=1}^{n}(z_j - \bar{z}_j)\right)$$

Bimatrix Potential Games

But since

$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} = \bar{y}_i = \sum_{j=1}^{n} z_j = \sum_{j=1}^{n} \bar{z}_j = 1$$

we conclude that

$$\sum_{i=1}^{m}(y_i - \bar{y}_i) = 0 \qquad \sum_{j=1}^{n}(z_j - \bar{z}_j) = 0$$

and therefore

$$y'Az - \bar{y}Az = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_i - \bar{y}_i)\phi_{ij}z_j = y'\Phi z - \bar{y}\Phi z$$

$$y'Bz - yB\bar{z} = \sum_{i=1}^{m}\sum_{j=1}^{n} y_i\phi_{ij}(z_j - \bar{z}_j) = y'\Phi z - y\Phi\bar{z}$$

which concludes the sufficiency proof.

Characterization of Potential Games

Characterization of Potential Games

II games: they have a function $\phi(\gamma)$ such that

$$J_i(\gamma) = \phi(\gamma), \quad \forall \gamma \in \Gamma, \quad i \in \{1, 2, \ldots, N\}$$

trivially satisfy the **(exact) potential game** condition, and therefore are potential games with potential ϕ.

Dummy games: outcome J_i of each P_i does not depend on the player's own policy γ_i

- but may depend on the policies of the other players, i.e.,

$$J_i(\gamma_i, \gamma_{-i}) = J_i(\bar{\gamma}_i, \gamma_{-i}) = J_i(\gamma_{-i}) \quad \forall \gamma_i, \bar{\gamma}_i \in \Gamma_i, \gamma_{-i} \in \Gamma_{-i}, i \in \{1, 2, \ldots, N\}$$

Dummy games are also potential games with constant potential

$$\phi(\gamma) = 0, \qquad \forall \gamma \in \Gamma$$

Characterization of Potential Games

The set of potential games is closed under **summation**.

Suppose that one has two games G and H with the same **action spaces** $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ but different outcomes for the same set of policies $\gamma \in \Gamma$

- G has an **outcome** given by $G_i(\gamma)$ for each player P_i, $\forall i \in \{1, 2, \dots, N\}$
- H has an **outcome** given by $H_i(\gamma)$ for each player P_i, $\forall i \in \{1, 2, \dots, N\}$

The **sum game** $G + H$ is a game with the same action spaces as G and H, but with the outcome

$$G_i(\gamma) + H_i(\gamma), \qquad \forall \gamma \in \Gamma$$

for each player P_i, $\forall i \in \{1, 2, \dots, N\}$.

Characterization of Potential Games

Proposition 12.4 (Sum of potential games).

If G and H are two potential games with the same action spaces and potentials ϕ_G and ϕ_H, then the sum game $G + H$ is also a potential game with a potential $\phi_G + \phi_H$

Corollary: the sum of an II game and a dummy game is always a potential game. Any potential game can be expressed as the sum of two games of these types.

Proposition 12.5. A game G is an exact potential game if and only if there exists a dummy game D and an II game H such that $G = D + H$.

Characterization of Potential Games

Proof of Proposition 12.5. Because of **Proposition 12.4**, if G can be decomposed as the sum of a dummy game D and an II game H, then G must be a potential game.

To prove the converse, show that if G is an exact potential game with potential ϕ_G, then we can find a dummy game D and an II game H such that $G = D + H$ (this is simple!). For the II game H, chose outcomes for all players equal to the potential ϕ_G:

$$H_i(\gamma) = \phi_G(\gamma) \qquad \forall \gamma \in \Gamma$$

This then uniquely defines what the outcomes of the dummy game D must be so that $G = D + H$:

$$D_i(\gamma) = G_i(\gamma) - \phi_G(\gamma), \qquad \forall \gamma \in \Gamma$$

Characterization of Potential Games

To check that this indeed defines a dummy game, we compute

$$D_i(\gamma_i, \gamma_{-i}) - D_i(\bar{\gamma}_i, \gamma_{-i}) = G_i(\gamma) - G_i(\bar{\gamma}) - \phi_G(\gamma) + \phi_G(\bar{\gamma})$$
$$\forall \gamma_i \bar{\gamma}_i \in \Gamma_i, \gamma_{-i} \in \Gamma_{-i}, \quad i \in \{1, 2, \ldots, N\}$$

which is equal to zero because ϕ_G is an exact potential for G.

This confirms that $D_i(\gamma_i, \gamma_{-i})$ indeed does not depend on γ_i.

Potential Games with Interval Action Spaces

Potential Games with Interval Action Spaces

Constructing a potential game is simple

- one only needs to add an II game to a dummy game.

For games whose action spaces are intervals in the real line, determining if a game is an exact potential game is simple.

Note: a matrix game with mixed policies and only two actions is an example of a game whose action spaces are intervals.

Lemma 12.1 (Potential games with interval action spaces)

For a game G, suppose every action space Γ_i is a close interval in $\mathbb{R}$ and every outcome J_i is twice continuously differentiable.

In this case, the following three statements are equivalent:

1. G is an exact potential game.

Potential Games with Interval Action Spaces

2. There exists a twice differentiable function $\phi(\gamma)$ such that

$$\frac{\partial J_i(\gamma)}{\partial \gamma_i} = \frac{\partial \phi(\gamma)}{\partial \gamma_i} \quad \forall \gamma \in \Gamma, \quad i \in \{1, 2, \ldots, N\}$$

when these equalities hold ϕ is an exact potential for the game.

3. The outcomes satisfy

$$\frac{\partial^2 J_i(\gamma)}{\partial \gamma_i \gamma_j} = \frac{\partial^2 J_j(\gamma)}{\partial \gamma_i \gamma_j} \quad \forall \gamma \in \Gamma, \quad i, j \in \{1, 2, \ldots, N\}$$

when these equalities hold, we can construct an exact potential using

$$\phi(\gamma) = \sum_{k=1}^{N} \int_0^1 \frac{\partial J_k(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_k} (\gamma_k - \zeta_k) d\tau \quad \forall \gamma \in \Gamma$$

where ζ can be any element of Γ.

Potential Games with Interval Action Spaces

Proof of Lemma 12.1.

To prove that **1** implies **2**, assume that G is an exact potential game with potential ϕ.

In this case, for every $\gamma_i, \bar{\gamma}_i, \gamma_{-i}$

$$J_i(\gamma_i, \gamma_{-i}) - J_i(\bar{\gamma}_i, \gamma_{-i}) = \phi(\gamma_i, \gamma_{-i}) - \phi(\bar{\gamma}_i, \gamma_{-i})$$

Dividing both sides by $\gamma_i - \bar{\gamma}_i$ and making $\bar{\gamma}_i \to \gamma_i$ we obtain precisely statement **2**

$$\frac{\partial J_i(\gamma)}{\partial \gamma_i} = \frac{\partial \phi(\gamma)}{\partial \gamma_i} \quad \forall \gamma \in \Gamma, \quad i \in \{1, 2, \ldots, N\}$$

Potential Games with Interval Action Spaces

To show that **2** implies **1**, integrate both sides of statement **2** between $\bar{\gamma}_i$ and $\hat{\gamma}_i$ and obtain

$$\int_{\bar{\gamma}_i}^{\hat{\gamma}_i} \frac{\partial J_i(\gamma)}{\partial \gamma_i} d\gamma_i = \int_{\bar{\gamma}_i}^{\hat{\gamma}_i} \frac{\partial \phi(\gamma)}{\partial \gamma_i} d\gamma_i, \quad \forall \bar{\gamma}_i, \hat{\gamma}_i, \gamma_{-i}, \quad i \in \{1, 2, \ldots, N\}$$

which is equivalent to

$$J_i(\hat{\gamma}_i, \gamma_{-i}) - J_i(\bar{\gamma}_i, \gamma_{-i}) = \phi(\hat{\gamma}_i, \gamma_{-i}) - \phi(\bar{\gamma}_i, \gamma_{-i}) \qquad \forall \bar{\gamma}_i, \hat{\gamma}_i, \gamma_{-i}, \quad i \in \{1, 2, \ldots, \}$$

proving that we have a potential game with potential ϕ.

We have shown that **1** and **2** are equivalent.

Potential Games with Interval Action Spaces

To prove that **2** implies **3**, take partial derivatives of both sides of statement **2** with respect to γ_j and conclude that

$$\frac{\partial^2 J_i(\gamma)}{\partial \gamma_i \gamma_j} = \frac{\partial^2 \phi(\gamma)}{\partial \gamma_i \gamma_j}, \qquad \forall \gamma, \quad i \in \{1, 2, \ldots, N\}$$

The right-hand side of the above equality does not change if we exchange i by j

Conclusion: the left-hand side cannot change if we exchange i by j.

This is precisely, what is stated in **3**.

To show that **3** implies **2**, we will show that the exact potential function defined in statement **3** satisfies statement **2**.

Potential Games with Interval Action Spaces

To this effect, we need to compute

$$\frac{\partial \phi(\gamma)}{\partial \gamma_i} = \frac{\partial}{\partial \gamma_i} \left(\int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} (\gamma_i - \zeta_i) d\tau \right.$$

$$\left. + \sum_{k \neq i} \int_0^1 \frac{\partial J_k(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_k} (\gamma_k - \zeta_k) d\tau \right)$$

$$= \int_0^1 \tau \frac{\partial^2 J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i^2} (\gamma_i - \zeta_i) d\tau + \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau$$

$$+ \sum_{k \neq i} \int_0^1 \tau \frac{\partial^2 J_k(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i \gamma_k} (\gamma_k - \zeta_k) d\tau$$

$$= \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau + \sum_{k=1}^{N} \int_0^1 \tau \frac{\partial^2 J_k(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i \gamma_k} (\gamma_k - \zeta_k) d\tau$$

Potential Games with Interval Action Spaces

Using outcomes of statement **3**, we conclude that

$$\begin{aligned}
\frac{\partial \phi(\gamma)}{\partial \gamma_i} &= \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau + \int_0^1 \tau \sum_{k=1}^{N} \frac{\partial^2 J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i \gamma_k} (\gamma_k - \zeta_k) d\tau \\
&= \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau \\
&\qquad + \int_0^1 \tau \left(\sum_{k=1}^{N} \frac{\partial}{\partial \gamma_k} \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} \frac{d(\zeta_k + \tau(\gamma_k - \zeta_k))}{d\tau} \right) d\tau \\
&= \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau + \int_0^1 \tau \frac{d}{d\tau} \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau
\end{aligned}$$

Potential Games with Interval Action Spaces

Integrating by parts, we finally obtain

$$\frac{\partial \phi(\gamma)}{\partial \gamma_i} = \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau + \left[\tau \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i}\right]_0^1 - \int_0^1 \frac{\partial J_i(\zeta + \tau(\gamma - \zeta))}{\partial \gamma_i} d\tau$$

$$= \frac{\partial J_i(\gamma)}{\partial \gamma_i}$$

which proves **2**.

At this point we have also shown that **2** and **3** are equivalent, which completes the proof.

Potential Games with Interval Action Spaces

Notation 4 (Potential games).

If we view the (symmetric of the) vector of the derivatives of the outcomes J_i with respect to the corresponding γ_i as a **force**

$$F := -\left(\frac{\partial J_1}{\partial \gamma_1}, \frac{\partial J_2}{\partial \gamma_2}, \cdots, \frac{\partial J_N}{\partial \gamma_N}\right)$$

that drives the players towards a (selfish) minimization of their outcomes, then condition **2** corresponds to a requirement that this force be conservative with a potential ϕ.

Recall: a mechanical force F is **conservative** if it can be written as $F = -\Delta\phi$ for some potential ϕ.

A mechanically inclined reader may construct potential games with outcomes inspired by conservative forces.

Practice Exercises

Practice Exercises

12.1.1 Consider a bimatrix game with two actions for each player defined by

$$A = \underbrace{\left.\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right\}}_{P_2 \text{ choices}} P_1 \text{ choices} \qquad B = \underbrace{\left.\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right\}}_{P_2 \text{ choices}} P_1 \text{ choices}$$

1. Under what conditions is this an exact potential game in **pure** policies?

Your answer should be a set of equalities/inequalities that the a_{ij} and b_{ij} need to satisfy.

Practice Exercises

Solution to 12.1.1. For this game to be a potential game, with potential

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

we need to have

$$a_{11} - a_{21} = \phi_{11} - \phi_{21} \qquad a_{12} - a_{22} = \phi_{12} - \phi_{22}$$

$$b_{11} - b_{12} = \phi_{11} - \phi_{12} \qquad b_{21} - b_{22} = \phi_{21} - \phi_{22}$$

which is equivalent to

$$\begin{aligned} a_{11} - a_{21} &= \phi_{11} - \phi_{21} \\ a_{11} - a_{21} - (b_{11} - b_{12}) &= \phi_{12} - \phi_{21} \end{aligned} \qquad \begin{aligned} a_{12} - a_{22} &= \phi_{12} - \phi_{22} \\ a_{12} - a_{22} - (b_{21} - b_{22}) &= \phi_{12} - \phi_{21} \end{aligned}$$

Practice Exercises

$$a_{11} - a_{21} = \phi_{11} - \phi_{21} \qquad a_{12} - a_{22} = \phi_{12} - \phi_{22}$$
$$a_{11} - a_{21} - (b_{11} - b_{12}) = \phi_{12} - \phi_{21} \qquad a_{12} - a_{22} - (b_{21} - b_{22}) = \phi_{12} - \phi_{21}$$

This system of equations has a solution if and only if

$$a_{11} - a_{21} - (b_{11} - b_{12}) = a_{12} - a_{22} - (b_{21} - b_{22})$$

in which case we can make, e.g.,

$$\phi_{22} = 0 \qquad \phi_{12} = a_{12} - a_{22}$$
$$\phi_{21} = \phi_{12} - a_{11} + a_{21} + b_{11} - b_{12} \qquad \phi_{11} = \phi_{21} + a_{11} - a_{21}$$

Practice Exercises

12.1.2 Show that the prisoners' dilemma bimatrix game defined by

$$A = \underbrace{\begin{bmatrix} 2 & 30 \\ 0 & 8 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} \; P_1 \text{ choices} \qquad B = \underbrace{\begin{bmatrix} 2 & 0 \\ 30 & 8 \end{bmatrix}}_{P_2 \text{ choices}} \Big\} \; P_1 \text{ choices}$$

is an exact potential game in pure policies.

Practice Exercises

Solution to 12.1.2. The potential for this game can be defined by the following matrix

$$\Phi = \underbrace{\left.\begin{bmatrix} 24 & 22 \\ 22 & 0 \end{bmatrix}\right\}}_{P_2 \text{ choices}} P_1 \text{ choices}$$

Indeed,

$$a_{11} - a_{21} = \phi_{11} - \phi_{21} = 2 \qquad a_{12} - a_{22} = \phi_{12} - \phi_{22} = 22$$
$$b_{11} - b_{12} = \phi_{11} - \phi_{12} = 2 \qquad b_{21} - b_{22} = \phi_{21} - \phi_{22} = 22$$

Practice Exercises

12.2. Consider a bimatrix game with two actions for both players defined by

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{P_2 \text{ choices}} \Big\} \; P_1 \text{ choices} \qquad B = \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{P_2 \text{ choices}} \Big\} \; P_1 \text{ choices}$$

1. Under what conditions is this an exact potential game in **mixed** policies?

Your answer should be a set of equalities/inequalities that the a_{ij} and b_{ij} need to satisfy.

2. Find an exact potential function when the game is a potential game.

Your answer should be a function that depends on the a_{ij} and b_{ij}.

Practice Exercises

3. When is a zero-sum game an exact potential game in mixed policies? Find its potential function.

Your answer should be a set of equalities/inequalities that the a_{ij} and b_{ij} need to satisfy and the potential function should be a function that depends on the a_{ij} and b_{ij}.

Practice Exercises

Solution to 12.2.1 Under the mixed policies

$$y := \begin{bmatrix} y_1 \\ 1 - y_1 \end{bmatrix}, \quad y_1 \in [0, 1] \qquad\qquad z := \begin{bmatrix} z_1 \\ 1 - z_1 \end{bmatrix}, \quad z_1 \in [0, 1]$$

for P_1 and P_2, respectively, the outcomes for this games are

$$J_1(y_1, z_1) = a_{11} y_1 z_1 + a_{12} y_1 (1 - z_1) + a_{21}(1 - y_1) z_1 + a_{22}(1 - y_1)(1 - z_1)$$
$$J_2(y_1, z_1) = a_{11} y_1 z_1 + a_{12} y_1 (1 - z_1) + b_{21}(1 - y_1) z_1 + b_{22}(1 - y_1)(1 - z_1)$$

and therefore

$$\frac{\partial^2 J_1}{\partial y_1 z_1} = a_{11} - a_{12} - a_{21} + a_{22} \qquad \frac{\partial^2 J_2}{\partial y_1 z_1} = b_{11} - b_{12} - b_{21} + b_{22}$$

$$\frac{\partial^2 J_1}{\partial y_1^2} = 0 \qquad \frac{\partial^2 J_2}{\partial y_1^2} = 0$$

$$\frac{\partial^2 J_1}{\partial y_1^2} = 0 \qquad \frac{\partial^2 J_2}{\partial z_1^2} = 0$$

Practice Exercises

In view of **Proposition 12.1**, we conclude that this is a potential game if and only if

$$a_{11} - a_{12} - a_{21} + a_{22} = b_{11} - b_{12} - b_{21} + b_{22}$$

Note that the prisoners' dilemma bimatrix game in **Example 12.1** satisfies this condition.

Practice Exercises

Solution to 12.2.2 According to **Proposition 12.1**, the potential function can be obtained by

$$\begin{aligned}
\phi(y_1, z_1) &= \int_0^1 \frac{\partial J_1(\tau y_1, \tau z_1)}{\partial y_1} y_1 + \frac{\partial J_2(\tau y_1, \tau z_1)}{\partial z_1} z_1 d\tau \\
&= \int_0^1 \left(a_{11}\tau z_1 + a_{12}(1-\tau z_1) - a_{21}\tau z_1 - a_{22}(1-\tau z_1)\right) y_1 \\
&\quad + \left(b_{11}\tau y_1 + b_{12}\tau y_1 + b_{21}(1-\tau y_1) - b_{22}(1-\tau y_1)\right) z_1 d\tau \\
&= \int_0^1 \left((a_{11} - a_{12} - a_{21} + a_{22})\tau z_1 + a_{12} - a_{22}\right) y_1 \\
&\quad + \left((b_{11} - b_{12} - b_{21} + b_{22})\tau y_1 + b_{21} - b_{22}\right) z_1 d\tau \\
&= \frac{1}{2}(a_{11} - a_{12} - a_{21} + a_{22}) y_1 z_1 + (a_{12} - a_{22}) y_1 \\
&\quad + \frac{1}{2}(b_{11} - b_{12} - b_{21} + b_{22}) y_1 z_1 + (b_{21} - b_{22}) z_1
\end{aligned}$$

Practice Exercises

Final result from **Example 12.2.1** states that

$$a_{11} - a_{12} - a_{21} + a_{22} = b_{11} - b_{12} - b_{21} + b_{22}$$

Then,

$$\begin{aligned}\phi(y_1, z_1) = \frac{1}{2}(a_{11} - a_{12} - a_{21} + a_{22})y_1 z_1 + (a_{12} - a_{22})y_1 \\ + \frac{1}{2}(b_{11} - b_{12} - b_{21} + b_{22})y_1 z_1 + (b_{21} - b_{22})z_1\end{aligned}$$

simplifies to

$$\phi(y_1, z_1) = (a_{11} - a_{12} - a_{21} + a_{22})y_1 z_1 + (a_{12} - a_{22})y_1 + (b_{21} - b_{22})z_1$$

Note that this is consistent with what we saw in **Example 12.1**.

Practice Exercises

Solution to 12.2.3 For a zero-sum game the left and the right hand sides of

$$a_{11} - a_{12} - a_{21} + a_{22} = b_{11} - b_{12} - b_{21} + b_{22}$$

are symmetric, which is only possible if

$$a_{11} - a_{12} - a_{21} + a_{22} = 0 \quad \Leftrightarrow \quad a_{11} + a_{22} = a_{12} + a_{21}$$

which corresponds to the outcomes

$$J_1(y_1, z_1) = a_{12}y_1 + a_{21}z_1 + a_{22} - a_{22}y_1 - a_{22}z_1 = -J_2(y_1, z_1)$$

In this case, a potential function is given by

$$\phi(y_1, z_1) = (a_{12} - a_{22})y_1 - (a_{21} - a_{22})z_1 = (a_{12} - a_{22})(y_1 - z_1)$$

End of Lecture

12 - Potential Games

Questions?